De Broglie's relativistic phase waves and wave groups

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Louis de Broglie's original theory of matter waves has until very recently received relatively little treatment in the literature on history and foundations of physics. This paper attempts to clarify a number of aspects of de Broglie's theory, as formulated primarily in his famous thesis, which still give rise to confusion or debate. To this end, a re-examination is made of the scope and validity of the relativistic physics employed in de Broglie's arguments concerning the existence of phase waves and their physical implications. Recent doubts concerning the soundness of de Broglie's introduction of a wave group associated with the quantum are also analyzed in detail, and it is argued that the wave group indeed has a problematic, if minor, role in the overall theory.

I. INTRODUCTION

Although considerable attention has been given within recent decades to the history of the development of quantum theory, relatively little has been written from a historical, and especially from an analytical, point of view concerning the fundamental contribution of Louis de Broglie in the early twenties. Important exceptions are the recent

works of MacKinnon[1] and Espinosa.[2] In this paper we shall take up a number of claims in these and related works which are in our opinion either questionable or insufficiently argued. The major points of discussion will be the claims that (i) de Broglie's theory of matter waves, although relativistic, only seemed to work in the nonrelativistic approximation (MacKinnon), (ii) the introduction of wave groups in the theory rests on a confusion between the relativistic phase waves and the constituent waves of the group (MacKinnon), and that (iii) de Broglie's derivation of the equality of energy transport and group velocity for the quantum is fundamentally sound (Espinosa). In all these cases, we shall argue for alternative points of view.

A resumé of the paper is as follows. De Broglie's fundamental relativistic postulates leading to the existence of phase waves for material quanta are reviewed in Sec. II. One well-known aspect of de Broglie's derivation of the phase waves is the appearance of an apparently irrelevant characteristic frequency associated with the moving quantum. Possible physical interpretations of this frequency are discussed, including one related to a little-known suggestion due to Gamow in 1926. In Sec. III, it is shown that, recent statements to the contrary notwithstanding, the relativistic version of the fundamental relation $p = h/\lambda$ is contained in de Broglie's thesis, although in a disguised form. Moreover, it is shown that this relation was not unique to de Broglie, but had been proposed independently (and before de Broglie's thesis) by Compton. De Broglie's demonstration that the group velocity of the phase waves is equal to the velocity of energy transport of the quantum is the subject of the next two sections. In Sec. IV, the notion of group velocity for a standard wave group is analyzed. It is argued that certain (spurious) conceptual problems have arisen regarding de Broglie's wave groups as a result of the fact that an elementary feature of the definition of group velocity went unnoticed in de Broglie's thesis, and has been similarly overlooked in recent commentaries. The debate concerning the more important question of the physical justification of de Broglie's wave groups is discussed in Sec. V. It is concluded that the wave groups are, within the context of the original theory, either fictitious, or otherwise introduced on the basis of *ad hoc* reasoning. In Sec. VI, a more coherent reconstruction of the de Broglie group model is suggested, and its compatibility (or otherwise) with the general features of de Broglie's program is discussed. Questions related to the physical applications of de Broglie's matter waves are the subject of the final sections of the paper. In Sec. VII, a re-examination of the derivation of the stability conditions for the Bohr hydrogen atom indicates that it should not be considered as based on purely classical considerations (although it does create problems for the wave-group model). Likewise, in Sec. VIII, it is recalled that experimental tests performed up to 1935 had already corroborated the relativistic version of the relation $p = h/\lambda$ in the case of electron diffracton. Finally, in Sec. IX, comments are made regarding the fate (and respective importance) of de Broglie's individual phase waves and wave groups up until the emergence of Schrödinger's comprehensive theory of wave mechanics.

II. THE KINEMATICAL PROPERTIES OF THE PHASE WAVES

In the period from 1923 to 1928, de Broglie proposed several different and conflicting approaches to his wave mechanics. We shall not describe all of them[3]; in this article we shall focus our attention on the formulation presented in his thesis.[4]

De Broglie endeavored to develop a unified theory for all kinds of physical entities. Electrons, protons, light quanta, etc; all of these should obey similar equations and correspond to similar pictures. All these physical entities were regarded by de Broglie as energy quanta, or energy atoms: uncuttable and unchangeable energy concentrations. Each kind of quantum—and in particular, the electron—was supposed to be characterized by its rest mass, equal for every quantum of a given kind. To the rest mass de Broglie associated the relativistic rest energy $E_0 = m_0c^2$.

In the classical electron theory, the mass is a consequence of the electromagnetic field. The field contains energy, and the rest energy (and therefore the rest mass) of the electron is therefore regarded as spread in space, and not concentrated in a small particle. De Broglie explicitly states: "What characterizes the electron as an energy atom is not the small spatial place that it fills—I repeat that it fills the whole space—it is the fact of being uncuttable, indivisible, it is its *unity*."[5]

The basic new quantitative hypothesis introduced by de Broglie is the association of an intrinsic frequency to each kind of quantum. The rest energy of the electron, m_0c^2, is associated with a proper frequency ν_0 through the quantum relation

$$m_0c^2 = h\nu_0. \tag{1}$$

The Planck–Einstein relation $E = h\nu$ was well known at that time, but it had a quite different meaning. De Broglie proposed a new generalization and interpretation of the equation,[6] and applied it to any kind of physical entity.

Now, joining all these elements, we have an initial picture of each quantum: it is a spatially extended pulsating indivisible energy. Notice that although the quantum is very large, it has a well-defined geometrical center, and the energy is strongly concentrated around it.

De Broglie supposed, for simplicity, that in the proper reference frame of the quantum, all of its parts pulsate with the same frequency described by Eq. (1), and that all points pulsate in unison, that is, with the same phase. A good analogy, in two dimensions, is provided by the simplest pulsating mode of a drum membrane: all points go up and down at the same time, although with different amplitudes.

This pulsating energy atom does not look like a traveling wave, in its rest system. If at this point we want to talk about waves, we can describe this pulsation as a wave of infinite speed and infinite wavelength, since the pulsation of the center of the quantum seems to propagate instantaneously to every point in space.[7] Or we can alternatively describe it as a standing wave, which may be mathematically decomposed into the superposition of an outgoing (expanding) and an ingoing (contracting) spherical wave. Indeed, in a paper published after his thesis, de Broglie proposes exactly this latter description for the quantum.[8]

It is well known that from this picture of the electron and from relativistic kinematics de Broglie was able to prove that for any observer that sees the quantum in motion with velocity v, the pulsation will be observed as a wave of velocity V

$$V = c^2/v \tag{2}$$

and frequency ν

$$\nu = \nu_0/(1 - v^2/c^2)^{1/2}. \tag{3}$$

Since the energy of the quantum also transforms according to a similar relation,

$$E = E_0/(1 - v^2/c^2)^{1/2}, \tag{4}$$

de Broglie was able to prove that the relation $E = h\nu$ may be coherently applied both to a quantum at rest and to a moving quantum.[9]

There is a point that has already been much discussed but that still deserves attention: if we transform the frequency of the quantum using the relativistic transformation for the frequency of a clock, we obtain, instead of Eq. (3), the following result:

$$\nu_1 = \nu_0(1 - v^2/c^2)^{1/2}. \tag{5}$$

This frequency ν_1 does not satisfy the relation $E = h\nu$. This does not need to bother us, since Eq. (5) is derived from the relativistic transformation for the frequency of a point-like clock, and the quantum is not pointlike; and the frequency transformation for waves (e.g., of light waves, in the Doppler effect) is different from the frequency transformation of clocks.

The new frequency ν_1 is not relevant in de Broglie's thesis, and we might consider it as devoid of interest. But this does not mean that this transformed rest frequency has no meaning or that it shows a weakness in de Broglie's theory. It has the obvious meaning of describing the frequency of oscillation of any point of space that moves with the same velocity v of the quantum, including its geometrical center.

There is also another interesting interpretation for ν_1 which is not described in recent works, and which was proposed by Gamow in 1926.[10] If instead of a single harmonic wave, we consider a wave group[11] built up of a set of phase waves of similar frequencies, this wave group will change its size in time, and will eventually spread.[12] But if the wave group is built of waves of very close frequencies, it will spread slowly. In this case, the prominent temporal effect, instead of a spreading, is a *pulsation* associated with the wave group. The envelope and size of the group will remain almost constant, but its fine structure will undergo periodical changes. The frequency of this phenomenon is

$$\nu' = \frac{\partial V(\lambda)}{\partial \lambda}, \tag{6}$$

where $V(\lambda)$ is the phase velocity V as a function of wavelength λ. For de Broglie's waves, we have[13]

$$V = (c^2 + \lambda^2\nu_0^2)^{1/2}, \tag{7}$$

and from Eqs. (6) and (7) we easily prove that

$$\nu' = \lambda\nu_0^2/V. \tag{8}$$

Since, for any wave, $V = \lambda\nu$, and also using Eqs. (3) and (5), we obtain

$$\nu' = \nu_1. \tag{9}$$

Hence, the frequency of pulsation of the wave groups of de Broglie's waves is equal to the transformed rest frequency of the quantum. This provides, in the case of wave groups, another physical interpretation of ν_1.

III. THE RELATIVISTIC MOMENTUM-WAVELENGTH RELATION

The main dynamical relation in all popular presentations of de Broglie's wave theory is the famous relation linking the momentum p of the quantum with the associated wavelength λ:

$$\lambda = h/p. \tag{10}$$

Since de Broglie's theory is essentially relativistic, we may also write this equation (in a field-free space) in this form:

$$\lambda = (h/m_0v)(1 - v^2/c^2)^{1/2}. \tag{11}$$

It has been recently claimed that this relation between wavelength and momentum appears only once in de Broglie's thesis, and then only as a nonrelativistic approximation[14]:

$$\lambda = h/(m_0v). \tag{12}$$

We do not agree with this statement, and will try to prove that the momentum-wavelength relation appears, in its relativistic formulation, but with different symbols, in de Broglie's thesis; and that the first explicit presentation of the relation $\lambda = h/p$ is due to Arthur Compton, not to de Broglie.

It is well known that de Broglie's relativistic theory does entail the relativistic relation (10). A very simple derivation is the following.

For any kind of wave, the phase velocity V is $V = \lambda\nu$; for de Broglie waves, the phase velocity V is related to the speed v of the quantum by the relation $V = c^2/v$; the frequency ν of de Broglie's wave is related to the energy E of the quantum by $E = h\nu$, and the relativistic energy E of the quantum is $E = m_0c^2/(1 - v^2/c^2)^{1/2}$. From these relations we derive both Eqs. (10) and (11). But this kind of derivation was not presented by de Broglie.

In his thesis, de Broglie presented a relativistic four-vector formulation of Fermat's principle, and related it to a similarly relativistic four-vector formulation of Hamilton's principle.[15] De Broglie described any wave by the covariant four-vector "Universe Wave" $\mathbf{O}$, with components:

$$O_1 = -(\nu/V)\cos(x, l), \quad O_2 = -(\nu/V)\cos(y, l),$$

$$O_3 = -(\nu/V)\cos(z, l), \quad O_4 = \nu/c. \tag{13}$$

In this set of equations, V is the phase velocity, and $\cos(x, l)$ etc. are the cosines of the angles between the direction of the ray associated with the wave, and the directions of the axes of the reference frame. De Broglie remarked:

> The [four] vector Universe Wave may be decomposed therefore into one time component $[O_4]$ proportional to the frequency and into a *space* vector $\mathbf{n}$ with the direction of propagation and with magnitude ν/V. We shall call this vector the "Wave number" because it is equal to the inverse of the wavelength.[16]

So, de Broglie defined the wavenumber vector as

$$\mathbf{n} = (O_1, O_2, O_3) \tag{14}$$

and he also stated that

$$|\mathbf{n}| = \nu/V = 1/\lambda. \tag{15}$$

Although de Broglie preferred to write ν/V instead of $1/\lambda$, it is evident that the three spacial relations of Eq. (13) could be written as $O_1 = -(1/\lambda)\cos(x, l)$, etc.

On the other hand, de Broglie had already defined the relativistic "Universe Impulse" covariant four-vector $\mathbf{J}$ as-

sociated with any particle. Its four components are

$$J_1 = -p_x, \quad J_2 = -p_y, \quad J_3 = -p_z, \quad J_4 = W/c, \tag{16}$$

where p is the momentum and W is the energy of the particle.

Using these four-vectors, de Broglie first remarked that the Einstein–Planck energy-frequency relation $E = h\nu$ may be written

$$O_4 = J_4/h \tag{17}$$

and then he proposed the generalization

$$O_i = J_i/h \quad (i = 1,2,3,4). \tag{18}$$

The space components of this relation may be written using the definitions contained in Eqs. (13) and (16):

$$(\nu/V)\cos(x, l) = p_x/h, \quad (\nu/V)\cos(y, l) = p_y/h,$$
$$(\nu/V)\cos(z, l) = p_z/h. \tag{19}$$

Since de Broglie had already stated that $\nu/V = 1/\lambda$, this set of equations describes the three space components of the relation

$$1/\lambda = p/h, \tag{20}$$

which is identical to Eq. (10). Hence, de Broglie's relation (18) is a fully relativistic covariant formulation that *contains* the relations $E = h\nu$ and $\lambda = h/p$.

It is curious that both MacKinnon and Espinosa reproduce Eq. (18), but they do not notice that it contains Eq. (10).

In the following pages of his thesis, de Broglie studied some special cases of these relations. He used, for instance,[17]

$$(m_0\beta c/h)/(1-\beta^2)^{1/2}\, dl = \nu/V\, dl, \tag{21}$$

where $\beta = v/c$. Using Eq. (15), we see that this equation is equivalent to Eq. (11).

At another page,[18] de Broglie also presented the relation

$$\mathbf{p} = h\mathbf{n}, \tag{22}$$

which is equivalent to Eq. (19) and is the vector form of Eq. (10). De Broglie also stated at another point[19]:

> We can, without loss of generality, take as the x-axis the direction of the motion of the system at the considered place, and call p_x the projection of the vector $\mathbf{p}$ in this direction. We have then the equation... $\nu/V = p_x/h$.

This relation is also obviously equivalent to Eq. (10). Instead of appearing only once, we see that the relation between the momentum and wavelength appears several times in de Broglie's thesis, and usually in its relativistic form. One does not easily detect these instances because instead of λ de Broglie uses the ratio V/ν; but this is just a change of notation, exactly as one may either write v or βc for the speed of the particle.

We may therefore state that de Broglie does present in his thesis the relativistic relation between momentum and wavelength, although with a different symbolism, and never under the form of Eq. (10).

The first author who explicitly ascribed to any moving system a wavelength, and wrote down the equation $p = h/\lambda$, exactly in this form, was Arthur Compton. This equation and its interpretation appeared in a paper presented by him in the Chicago meeting of the American Physical Society, on 1 December 1923 (one year *before* the presentation of de Broglie's thesis). This paper has never been published in its full form, but an abstract containing the relevant information appeared shortly after the meeting, in the *Physical Review*.[20] In the Appendix we reproduce the whole abstract since this important independent proposal of the momentum-wavelength relation has never been pointed out.[21]

Compton used Sommerfeld's quantum rule $\int p\, dq = nh$, which had been first used for periodic motion (such as electron orbits in the hydrogen atom) and makes the bold step of applying it to a rectilinear motion. Since this kind of motion does not seem to be periodical, Compton introduces or postulates a periodicity, and is therefore led to interpret any moving system as a wave. From Sommerfeld's rule, he derives $p = h/\lambda$.

Compton also uses the relation $p = E/V$, which was already known to apply to any kind of wave,[22] and shows the compatibility of the relations $E = h\nu$ and $p = h/\lambda$. Since all his assumptions are valid also in the theory of relativity, his relation $p = h/\lambda$ is relativistic; but it is not clear whether he noticed this aspect of his theory.

Compton's ideas had little impact; and he never claimed priority for the momentum-wavelength equation. But the fact remains that the first explicit presentation of this equation is due to him.

IV. WAVE GROUPS AND GROUP VELOCITY

In his first 1923 paper on phase waves, de Broglie explicitly refers to these entities as "fictitious."[23] In his next paper, however, they are termed merely "non-material"; the clarification of their physical significance being the "difficult task of an enlarged [theory of] electromagnetism."[24] The most detailed examination of their physical status is found in the thesis, written after the publication of these papers. Here, de Broglie offers a "gross" mechanical analogy[25] for the quantum, based on a horizontal, circular platform from which are suspended a number of equal weights on identical springs, all oscillating in phase. The distribution of the weights is nonuniform, being densest at the middle of the platform. An observer in horizontal motion with velocity $v = \beta c$ with respect to the platform will see the weights out of phase, and a given phase point will define a sinusoidal wave with velocity c/β. This obviously corresponds in the analogy to the phase wave of the quantum. De Broglie's explicit purpose in introducing this "lengthy" analogy is to show that the "phase" velocity c/β for the mechanical setup is clearly not the velocity of energy transport (which is v), although it is a real, observable effect due to the oscillations of the springs and the relative motion of the observer.

De Broglie had, of course, recognized in his early work that the superluminal speed c/β of the phase wave cannot signify energy transport. His full solution of this problem was the well-known demonstration that if the quantum is associated with a *group* of phase waves, the group velocity of the disturbance is equal to the velocity v of the quantum. The fact that this relation is derived essentially from the dispersion relation for the vacuum, which is in turn derivable from de Broglie's basic postulates, gives to the general scheme a strong claim to consistency or at least physical coherence. Owing to the fact that the derivation is, or is believed to be, devoid of *ad hoc* elements, this result is considered one of the strong points of the theory. Nevertheless, there has recently been some debate as to the cogency of de Broglie's original reasoning in this respect. Before turning to an examination of the question (which will be underta-

ken in Sec. V), it may be useful first to clear up several preliminary points regarding the role of the group velocity notion in de Broglie's theory.

De Broglie showed in his thesis that the group velocity (or velocity of beat propagation) U of a wave disturbance created by the superposition of two sinusoidal traveling waves of slightly different frequencies in a dispersive medium is given by[26]

$$\frac{1}{U}=\frac{d(\nu/V)}{d\nu}, \tag{23}$$

where $\delta\nu \ll \nu$. He then applied this formula (without justification) to the case of a wave group formed by the superposition of an *unlimited* number of phase waves whose frequencies vary continuously within the interval ν, $\nu+\delta\nu$. De Broglie then showed, using the equations $V=c/\beta$, $\nu=m_0c^2(1-\beta^2)^{-1/2}/h$ for the relativistic phase waves, that

$$U=\beta c=v. \tag{24}$$

He then remarked: "... in the wave theory of dispersion, if one excludes the regions of absorption, the energy velocity is equal to the group velocity. Even though we examine it from a very different point of view, we find an analogous result since the velocity of the moving body is nothing but the energy displacement velocity."[27]

This important shift in de Broglie's reasoning, from a single monochromatic wave representing the quantum to a wave group, has not always been regarded by students of the theory as a completely straightforward one. Even today, one occasionally encounters such questions as this. How can we attribute a single wavelength λ to the quantum when it is represented by a superposition of waves of many different wavelengths?[28] Another version of what is basically the same question is as follows. For the wave group, the intrinsic indeterminacy in λ, along with the relation $p=h/\lambda$, gives rise to a corresponding indeterminacy in the momentum p of the quantum. How then does the group velocity come out having a definite value?

As it happens, this arguably minor, but nevertheless real problem, need not have greatly bothered de Broglie, as in the thesis he explicitly *postulated* an indeterminacy in the velocity of the quantum. (This step constituted what has become a controversial aspect of de Broglie's reasoning in relation to wave groups, and will be examined in Sec. V. The remaining analysis in this section does not strictly follow his reasoning in the thesis in this sense.) However, in modern discussions of de Broglie's work, the velocity v is frequently treated as fixed, at least implicitly.[29] In such cases, the problem stands. Nevertheless, there is a straightforward solution within the general lines of de Broglie's scheme (we need not appeal to the subsequent treatment in quantum mechanics, as some would have it[28]).

If we introduce (using standard modern notation) a wave group as a superposition of an unlimited number of harmonic waves with wavenumber and angular frequency spreads $\Delta k, \Delta\omega$, respectively ($k=2\pi/\lambda$, $\omega=2\pi\nu$), then in those situations where the group velocity is well defined, we have[30]

$$U=\left(\frac{d\omega}{dk}\right)_{k=k_0}, \tag{25}$$

where k_0 is the center of the band Δk. For a dispersive medium, $\omega=\omega(k)$, $V=V(k)$, so from the de Broglie relation $V=c^2/v$ [Eq. (2) above], it is clear that v must also be a function of k. (In fact, from the relation $E=\hbar\omega$, the energy and hence the relativistic mass m of the quantum also have small indeterminacies related to $\Delta\omega$. We shall discuss a possible implication of this point in Sec. V.) Using $V=\omega/k$, we have

$$v=c^2k/\omega, \tag{26}$$

whence

$$\Delta v\sim\frac{dv}{dk}\Delta k=c^2\frac{d}{dk}\left(\frac{k}{\omega}\right)\Delta k. \tag{27}$$

For de Broglie's theory to be consistent, there should be a corresponding spread in U, as defined in Eq. (25). In fact, it is readily seen from the standard derivation of the group velocity that the original spread in k does give rise to a spread in U, given by[31]

$$\Delta U\sim\frac{d^2\omega}{dk^2}\Delta k. \tag{28}$$

Now the (dispersion) relation between ω and k can be easily derived from the relations $E=\hbar\omega$, $p=\hbar k$, and is[32]

$$\omega=[c^2k^2+(m_0c^2/\hbar)^2]^{1/2}. \tag{29}$$

Solving for $d\omega/dk$ in Eq. (29), and substituting the result in Eq. (28), we obtain

$$\Delta U\sim c^2\frac{d}{dk}\left(\frac{k}{\omega}\right)\Delta k. \tag{30}$$

Comparing this result with Eq. (27), the desired consistency result is obtained:

$$\Delta U\sim\Delta v. \tag{31}$$

To further demonstrate the consistency of de Broglie's formalism in this respect, if one solves Eq. (30) for ΔU, using $\omega=E/\hbar=mc^2/\hbar$, the result is

$$\Delta U\sim(\hbar/m)(1-\beta^2)\Delta k. \tag{32}$$

Putting $\Delta k=\Delta p/\hbar$, one immediately obtains

$$\Delta U\sim[(1-\beta^2)/m]\,\Delta p. \tag{33}$$

Now in the classical approximation ($\beta\to 0$), this again gives rise to the pre-established result $\Delta U\sim\Delta v$. In the general case, Eq. (33) may at first sight appear at odds with this result. However, it must be recalled that in this case,

$$\Delta p=m\Delta v+v\Delta m. \tag{34}$$

We have

$$\Delta m\sim\frac{dm}{dv}\Delta v=\frac{mv/c^2}{(1-\beta^2)}\Delta v. \tag{35}$$

From Eqs. (34) and (35), we get

$$\Delta p\sim m\Delta v/(1-\beta)^2. \tag{36}$$

Substituting this expression for Δp in Eq. (33), $\Delta U\sim\Delta v$ is once again derived.

The fact that there is an intrinsic indeterminacy in the group velocity also clarifies what may be considered at first sight another curious, if again relatively minor, aspect of some modern rederivations of de Broglie's result $U=v$. For a wave group with spreads $\Delta\omega, \Delta k$ the group velocity is rigorously given by Eq. (25), $U=(d\omega/dk)_0$, but in solving for U on the basis of Eq. (29) above, the identity $U=d\omega/dk$ [equivalent to de Broglie's expression in Eq. (23) above] is frequently used,[33] even though $d\omega/dk$ is a function of k in de Broglie's theory and hence not a fixed value. Curiously,

if one erroneously considers the velocity v as fixed (as is sometimes the case),[34] one *has* to use $U = d\omega/dk$ to derive the result $U = c^2k/\omega = c^2/V = v$. For to use the more correct $U = (d\omega/dk)_0$ leads to

$$U = c^2k_0/\omega(k_0) = c^2/V(k_0),$$

and it is not immediately clear that this is equivalent to v.[35] However, once the necessary existence of a finite Δv, coupled with the fact that $\Delta U \sim \Delta v$, is recognized in the theory, it is readily seen that either of the above two expressions for U is adequate for the problem at hand.

V. DE BROGLIE'S JUSTIFICATION OF THE WAVE GROUP MODEL

Let us now return to the more fundamental question of the status of de Broglie's wave groups within his theory of phase waves. MacKinnon has argued that the introduction of wave groups in the thesis rests on a basic confusion between two quite different concepts: the harmonic phase waves making up the wave group and the original relativistic phase waves.[36] MacKinnon's arguments to this end are, in our view, somewhat unclear, and one deficiency in particular has been pointed out by Espinosa.[2] We shall argue, however, that MacKinnon is basically correct in questioning the coherence of de Broglie's remarkable use of wave groups in the thesis, although our arguments do not entirely coincide with his.

De Broglie justified the introduction of a group velocity in his thesis in the following fashion. "If one attributes to the particle a velocity $v = \beta c$ without giving to β a completely determined value, but imposing on it only that it be between β and $\beta + \delta\beta$, the frequencies of the corresponding waves fill a small interval ν, $\nu + \delta\nu$."[37] This statement follows the discussion of the beat phenomenon obtained by superposing two harmonic waves of slightly different frequencies in a dispersive medium, which we referred to in Sec. IV. Thus it is clear that when de Broglie introduced the indeterminacy $\delta\nu$, he was thinking of a real superposition of phase waves whose frequencies lie in the interval ν, $\nu + \delta\nu$. The interesting point here is that the spread in frequencies $\delta\nu$ is an outcome of the postulated indeterminacy in the velocity v of the quantum.

MacKinnon's central criticism of de Broglie's introduction of wave groups is essentially that the "... relativistic phase waves do not involve dispersion while wave packets do ... ."[38] According to MacKinnon, there is no dispersion in the rest system, and no dispersion is introduced by the Lorentz transformation to any other system. "De Broglie's identification of the relationship of particle and relativistic phase velocity with the relationship between wave and group velocity is simply wrong. It rests on a conflation of two radically different concepts."[38] Moreover, MacKinnon argues that the de Broglie physics of wave groups is essentially classical, so that the frequency $\nu = E/h$ is being surreptitiously assigned two quite different meanings. (When dealing with wave groups, E is the classical energy, whereas in the treatment of phase waves, E contains the relativistic mass energy.)

One difficulty in MacKinnon's criticism, or rather criticisms, had to do with his somewhat misleading use of the term "dispersion" for the phase waves. Dispersion is strictly speaking a property of the medium, not the *individual* phase waves, and de Broglie's medium (empty space) is certainly dispersive for his matter waves, as Espinosa pointed out.[39] Equation (29) above is precisely the (nonlinear) dispersive relation for the medium. (And it is obviously relativistic, so that there is no confusion in the meaning of the frequency, as MacKinnon claims.)

It may be thought, however, that the quibble here is in part terminological, and that what MacKinnon really meant by saying that the phase wave does not disperse is that there is no frequency spread to be found anywhere in de Broglie's original arguments in favor of the existence of the relativistic phase wave. In fact, MacKinnon makes this point explicitly in his article,[38] and it is correct. (We shall come back to this point shortly.) But there are two problems in this reading of MacKinnon's critique. First, if this is really what he means when he says that the phase waves are nondispersive, then why does he shortly afterwards remark that in his later writings, de Broglie "tried to handle this difficulty by defining $n = (1 - \nu_0^2/\nu^2)^{1/2}$ [n is the refractive index] as the dispersion of space"?[40] This remark only makes sense if "dispersion" of the phase waves is understood in the standard fashion. Second, MacKinnon does not acknowledge that after introducing the relativistic phase waves, de Broglie explicitly *derives* $\delta\nu$ in his thesis as an outcome of the *postulated* indeterminacy in $\delta\beta$.

Thus the real quesiton is what is going on physically when wave groups are justified on the basis of the postulated indeterminacy $\delta\beta$ in de Broglie's thesis. In his recent criticism of MacKinnon, Espinosa finds no difficulty here.

> A particle with velocity v can be given arbitrarily small increments of velocity making consequent changes in the wave number, making it possible to define a group velocity for the particle. What is involved here is similar to the process of evaluating the derivative of a function at a point and contrasting this with the fact that a derivative cannot be defined for a function of an isolated point; it is necessary that values of the function exist over a neighborhood of points surrounding the point of interest. Only if all particle velocities defined the same wave-number phase wave would it be true that neither a wave packet nor a group velocity could be defined for a single particle. But such is not the case here ... . Therefore, de Broglie's procedure of giving the particle a virtual change in velocity and then leading to [the equation $U = v$] is justified.[41]

In our view, however, Espinosa's mathematical analogy of a derivative of a function at a point, although correct, is quite irrelevant to the fundamental issue at hand. What is the physical origin of the incremental changes in v? If these changes are "virtual," then is not the wave group thus produced also a "virtual" phenomenon?

Since we have been dealing up to this point with a free electron, there is no physical agent which alters the velocity of the quantum. There can then be only two possible interpretations of the indeterminacy $\delta\beta$. The first is that $\delta\beta$ reflects the degree of contingent ignorance on the part of the moving observer of the velocity of the quantum (interpretation A). The second is that $\delta\beta$ represents a real, *intrinsic* fuzziness or indeterminacy in this velocity (interpretation B).

The problems with interpretation A are several and fairly obvious. To begin with, it is still not at all clear where the physical wave *group* comes from—in reality we still have a single phase wave associated with the quantum whose precise frequency is unknown, and not a superposition of such waves. We cannot manufacture phase waves (which, as we saw earlier, are real if still obscure phenomena for de Brog-

lie) via ignorance! The putative wave group which results from $\delta\nu$ is nothing more than a fictitious ignorance-wave disturbance with no physical counterpart. Moreover, the characteristics of the wave group (except its group velocity) are not related to objective properties of the quantum nor of the medium: more or less information on the part of the observer would presumably make $\delta\beta$ and hence $\delta\nu$ as small or large as we like (consistent with the constraint $\delta\nu \ll \nu$), so that the dimensions of the group depend on other than physical considerations. All of this makes extremely hollow the claim that the group velocity can be interpreted as the velocity of the energy transmission of the quantum.

A second problem arises from the notion of superposition of an unlimited number of phase waves of slightly varying frequencies. If, as we saw in Sec. II, de Broglie associates to every energy a frequency, one wonders if one should not also associate to every frequency an energy. Then, if the wave group is a real, and not merely a fictitious superposition of phase waves, as de Broglie seems to suggest, it is not clear how to avoid the unpleasant conclusion that the energy of the group should be unlimited. De Broglie does not discuss this question in his thesis, nor in his preceding papers.

Interpretation B, on the other hand, smacks of the modern quantum theoretical view espoused primarily in some variants of the Copenhagen interpretation.[42] But it is totally without theoretical justification in the context of de Broglie's thesis and his other works of the period. There is not the slightest hint in de Broglie's theory of relativistic phase waves (Sec. II above) that the velocity of the quantum suffers an intrinsic fuzziness (even if small), nor would this make any sense in the relativistic physics that he employs. Interpretation B, if taken seriously, would imply an *ad hoc* and incongruous step in de Broglie's reasoning. And there is no textual basis for the belief that de Broglie espoused this interpretation.

In conclusion, we agree with MacKinnon that in the thesis, de Broglie's introduction of wave groups, and the consequent resolution of the problem of the superluminal velocity of the phase wave, is far from coherent, although not wholly for the same reasons.

VI. A REINTERPRETATION OF THE DE BROGLIE WAVE GROUP

A possible reconstruction and defense of de Broglie's result $U = v$ might be the following. Wave groups are simply to be postulated *ab initio* as representing the real properties of the quantum, with the component phase waves having a clearly secondary theoretical function—it is the group that is real, not the individual phase waves. (This would obviously avoid the problem of justifying the existence of wave groups when one starts, as does de Broglie, with a single phase wave whose characteristics are fixed for the quantum.) But this version of the theory would go a long way towards robbing de Broglie's thesis of its most important results. As MacKinnon[43] has effectively pointed out, the major applications therein of the matter-wave theory, viz., the explanations of the quantum rules for stable electron orbits and of the equilibrium properties of the quantum gas, and the new derivation of Planck's law for blackbody radiation, rely on the properties of the individual phase waves and not of the wave group.[44] (We shall consider the first of these applications in more detail in Sec. VI.)

In what follows, we outline another, more plausible reinterpretation, on roughly similar lines, that does not suffer from this objection.

De Broglie introduced some kind of indeterminacy associated with the quantum in order to derive his theorem $U = v$. But we may recall that wave groups are a completely classical phenomenon in the theory of sound or in electromagnetic theory; and since these theories are completely deterministic, we may ask whether there is not a deterministic way of accounting for wave groups in de Broglie's theory, in order to avoid the interpretation problems discussed in Sec. V.

We have already seen that in de Broglie's thesis each quantum is described as an energy atom which occupies a large region of space, and that this energy is concentrated around a central region. Hence, in the rest frame, instead of describing the pulsation of the electron by an equation, such as

$$u = A \cos 2\pi(\nu_0 t + \alpha), \tag{37}$$

with a constant amplitude A, a more realistic description would be provided by something such as

$$u = f(x, y, z) \cos 2\pi(\nu_0 t + \alpha). \tag{38}$$

Transforming the space-time coordinates to another reference frame, with the use of Lorentz transformations, we obtain

$$u = f\left(\frac{x' - vt'}{(1 - v^2/c^2)^{1/2}}, y', z'\right)\cos 2\pi\left(\nu_0 \frac{t' - x'v/c}{(1 - v^2/c^2)^{1/2}} + \alpha\right)$$

$$= f'(x' - vt', y', z')\cos 2\pi[\nu(t' - x'/V) + \alpha], \tag{39}$$

where we have made use of Eqs. (2) and (3).

This new equation may be interpreted as the description of a periodic wave of frequency ν and velocity v, modulated by an amplitude function f'. This amplitude moves with a velocity v.

Now, although in the above equation we have only one frequency ν, a modulated wave is not considered to be a monochromatic wave. It may instead be mathematically described as a superposition of a set of infinite, strictly monochromatic waves of different frequencies. Notice that this is not a quantum property: it is just a standard concept of classical wave theory.

Hence, the moving quantum may be mathematically described either as having a single frequency ν, but a modulated amplitude; or as a superposition of a set of unmodulated waves of different frequencies. If we opt for the latter approach, the quantum can be *described* as a wave group, and the introduction of the group velocity in de Broglie's theory becomes acceptable: since these harmonic unmodulated waves (Fourier components) are just mathematical constructs, we need not ascribe to them a momentum and energy. The momentum and energy of the quantum would be related to the wavelength and frequency (respectively) of the unique physical *modulated* wave, and hence there will be no indeterminacy in these magnitudes.[45] This interpretation solves the problems discussed in Sec. V (and removes the suspicion that the energy of the wave group is unlimited).

Now although this is a physically sound and coherent justification of the introduction of group velocity in de Broglie's theory, it is fairly clear that de Broglie did not have this view in mind in writing the thesis. However, it is interesting to compare this interpretation with a passage from one of de Broglie's "scientific poems," written some

years after the thesis, and published for the first time in 1982. In a note concerning the meaning of the Heisenberg uncertainty relations, de Broglie wrote the following:

> It is... essential to remark that if we consider a wave train, mathematically represented by a superposition of Fourier components, it is the train alone which has physical reality. The Fourier components exist only in the theoretician's mind.[46]

VII. DE BROGLIE ON THE STABILITY OF THE BOHR ATOM

The first and most striking application of de Broglie's theory of phase waves was, of course, the case of the Bohr atom. MacKinnon[47] has correctly pointed out that de Broglie's first derivation of Bohr's mysterious quantization rule for the hydrogen atom differs from that found in his thesis. In his first 1923 paper on phase waves, de Broglie derived the angular momentum quantization rule on the basis of a putative resonance effect, where the phase wave is required to be in phase with the intrinsic vibration of the electron. In the thesis, he introduced a general resonance condition for the phase waves themselves, and shows how this is related to constraints on wavelengths in certain cases. We shall now consider the derivation found in the thesis.[48]

The electron moves in a circular orbit of radius R around the nucleus with angular velocity $\omega = v/R$, where v is the tangential velocity (ω should not be confused with the angular frequency of the phase waves). Those orbits are stable for which the following resonance condition holds: $l = n\lambda \quad (n = 1,2,3,...)$, where l is the path length (in this case the orbit circumference $2\pi R$) when the wavelength is constant,[49] and $\int (\nu/V)dl = n$ in the general case. For reasons of elegance, de Broglie chose to use the latter condition (it establishes a link with the preceding discussion of Fermat's principle in the thesis), but for the case of the Bohr hydrogen atom, it is equivalent to the simpler condition on wavelengths. Using the relations $h\nu = m_0c^2$ and $V = c^2/v = c^2/\omega R$, de Broglie immediately solves the integral and obtains

$$m_0\omega R^2 = n\hbar, \tag{40}$$

which is effectively (i.e., ignoring the distinction between relativistic and classical mass) Bohr's quantization rule for angular momentum.

It is worth making several observations here. First, de Broglie's reasoning is essentially classical. No distinction is made between the intrinsic rest frequency of the electron and the frequency of the moving phase wave relative to the observer stationary with respect to the nucleus. The use of relation $h\nu = m_0c^2$ is, however, relativistic. This certainly introduces an incongruity into the reasoning. (MacKinnon[50] has criticized the derivation along these lines, although again his arguments differ somewhat from our own.) However, in his earlier 1923 derivation de Broglie *had* given a purely relativistic treatment, showing explicitly that Bohr's rule follows for small β.[51] Thus, if he was aware of this incongruity in his thesis, he may not have been particularly upset by it. In fact, if one does the above derivation relativistically (which in the thesis de Broglie does not), one obtains

$$m_0\omega R^2 = n\hbar(1-\beta^2)^{1/2}, \tag{41}$$

which clearly reduces to Bohr's rule for small β.[52]

A second observation is that on the basis of the resonance condition alone [$\lambda = 2\pi R/n$, or equivalently the relation (40)], it would in principle be possible for a given stable orbit to be associated with any (discrete) number of wavelengths, and a given wavelength to be associated with any (discrete) number of stable orbits. It is, of course, the remaining dynamical equilibrium conditions in Bohr's semiclassical theory that imply [in conjunction with relation (40)] that the R's form a discrete set—the same for all hydrogen atoms—and that there is a *unique* λ associated with each stable orbit. Thus the circumference of the nth orbit will be exactly equal to $n\lambda_n$, where λ_n is the wavelength of the phase wave associated with the electron in that orbit. It is perhaps curious that de Broglie did not state this explicitly in the thesis: he may have considered it obvious, although it is not a trivial outcome of his resonance condition on orbits alone.

An immediate outcome of this result is the following. If we imagine the electron in a given stable orbit to be associated with a more or less localized *group* of superposed phase waves, where $U = v$, then at most one of the constituent waves can exactly obey the resonanace condition for that orbit and the condition for dynamical equilibrium. If one argues—and it would seem reasonable to do so—that in the group representing the electron there are no grounds in the theory for assigning one particular constituent wave a privileged status, then one would conclude that the wave group model does not fit in well with the Bohr atom. Moreover, this objection can be strengthened by recalling the fact that in Bohr's hydrogen atom, the velocity of the electron is fixed for every stable orbit. Hence there is no obvious spread in β (whatever that means) and thus de Broglie's justification (see Sec. V) of the wave group is again clearly questionable in this case.

It appears, however, that de Broglie himself did not consider his theory of the Bohr atom as incompatible with the wave-group model. In fact, in a seldom cited 1924 article,[53] de Broglie explicitly allows for the orbital electron to be represented by a superposition of phase waves, only one of which exactly obeys the resonance condition for the orbit in question. The object of this paper was to explain a further result of Bohr, viz., for large n, there is a relationship between the frequency of emitted radiation and the angular velocity of the electron. The details of de Broglie's 1924 theory are extremely curious, and deserve a separate analysis. Suffice it to say here that at least in 1924, de Broglie did not consider his account of the Bohr atom incompatible with wave groups; on the contrary, superposition of phase waves are considered necessary for understanding the radiation emission properties of atoms with large quantum numbers (and eventually those of low quantum numbers as well). It is interesting that details of this theory are not included in the thesis.

VIII. EXPERIMENTAL VERIFICATION UP TO 1935

It is occasionally remarked that the only successful applications of de Broglie's original ideas, including the phenomenon of electron diffraction, were in the nonrelativistic domain. This view has naturally given rise to the query: why should a theory based on relativistic considerations only work in this domain?[54] To show that this query is misguided, we shall now briefly examine the nature of the electron diffraction experiments in the decade following de Broglie's thesis.

The first experimental evidence concerning the existence of waves associated with electrons was unwittingly found before de Broglie's theory, in 1921, by Davisson and Kunsman.[55] A connection between these results and de Broglie's theory was made by Elsasser in 1925,[56] who correctly interpreted the empirical results. These experiments, where reflection of electrons from a crystal face was observed, were repeated by Davisson and Germer.[57] They employed electrons of very low speed, accelerated by a potential difference smaller than 100 V. The agreement with de Broglie's theory was very poor, with differences of up to 30%. As pointed out by Bethe,[58] this difference was probably due to a variation of the speed of the electrons (and consequently a change of wavelength) inside the crystal.

This problem was avoided by Thomson, who worked with electron transmission through thin films.[59] Preliminary results were published by Thomson and Reid in 1927,[60] and the experiment was later carried out with several different material films and a refined technique by Thomson[61] and his collaborators.[62]

The ratio between the wavelengths as calculated from the classical formula $\lambda_c = h/(m_0 v)$ and from the relativistic formula $\lambda_r = h/mv$ is

$$\lambda_c/\lambda_r = [1 + eP/(2m_0c^2)]^{1/2}, \tag{42}$$

where P is the accelerating potential difference, e is the charge of the electron, and m_0 its rest mass. The difference between the classical and relativistic wavelengths is therefore proportional to P for low voltages. Thomson's group worked with electron energies of up to 70 000 eV. Thomson always used the relativistic relation in his computations of the relation between momentum and wavelength, but remarked that the relativistic effect amounted to only about 3%.[63] Since in his first set of data the agreement between theory and experiment was about 5%, these experiments could not discriminate between the classical and relativistic relations.

In the later experiments of the Thomson group, a better agreement between the experimental data and the relativistic prediction was reached, with differences of only 1%; but since the relativistic effect was of the same order of magnitude as the error, this was a poor verification of the relativistic correction.

Hence, if these had been the sole experiments on electron diffraction, it would be correct to say that only the nonrelativistic relation $\lambda_c = h/(m_0 v)$ had been verified. But besides the work of Davisson and Thomson and their respective groups, several new experiments were performed in the following years.[64] Although they are not usually cited in the literature, they are of considerable interest for our discussion.

In 1928, Kikuchi was able to reach an agreement of 0.2% between the relativistic theory and his experimental data for electrons of an energy up to 78 000 eV. The error is 15 times smaller than the relativistic correction.[65]

In 1931 Rupp reported the obtaining of diffraction with electrons of energies between 100 and 250 keV, and two years later Kosman and Alichanian studied diffraction patterns of electrons with energies up to 500 keV.[66] All of these results were in agreement with the relativistic prediction, but the most convincing results emerged in 1935 when Hughes studied electrons with energies of up to 1 MeV.[67] In Hughes's experiments, the difference between the relativistic and classical wavelengths reached 40%, and he found that his data agreed with the relativistic prediction within 5%.

We may conclude that between 1927 and 1935 several experiments using electrons of increasing energies produced very good accumulated confirmation of de Broglie's *relativistic* relation $\lambda = h/p$. Clearly, the results of these experiments are incompatible with the classical relation $\lambda = h/(m_0 v)$.

IX. SUBSEQUENT DEVELOPMENTS AND FINAL COMMENTS

As mentioned in Sec. V, it is the individual phase wave, and not the wave group, that is employed in the thesis in the applications of de Broglie's theory to the phenomena of atomic electrons, the quantum gas, and blackbody radiation. With the exception of the above-mentioned 1924 theory of atoms with large quantum numbers (which gives every indication of being an opportunistic theory that was quickly dropped), the introduction of wave groups in de Broglie's work serves a single purpose: to clarify the question of the superluminal velocity of the phase wave. Once this problem was solved, de Broglie simply ignored other aspects of the wave group model that were later to become of great concern in Schrödinger's work, e.g., the strict size of the group and the problem of its spreading in time as a result of the dispersive nature of the vacuum. It is important also to observe in this respect that in the subsequent experimental verifications of de Broglie's matter waves, essentially what is measured, as we have seen above, is the wavelength of the phase wave. The experimental results neither require nor suggest the existence of a wave group.

It is possible then that the reason little (known) criticism was raised by his early readers in relation to de Broglie's wave groups, besides the historical fact that de Broglie's ideas quickly gave way to Schrödinger's more comprehensive developments, is that they played a relatively restricted role in the overall theory. Moreover, the result $U = v$ is a highly nontrivial and satisfactory one, so readers may have felt (justifiably) that the wave group model was at least conceived on the right lines.

The role played by Einstein in the divulgation of de Broglie's work, and particularly in stimulating (or otherwise) Schrödinger's interest in matter waves, has been discussed in a number of places,[68] and we shall not repeat the details here. Interestingly, several commentators have speculated that prior to reading the thesis, Einstein himself advocated some kind of matter wave theory.[69] At any rate, when Einstein applied de Broglie's ideas in his second paper on ideal (Bose) gases,[70] he used, as MacKinnon pointed out, "only the general idea of associating waves with matter, and not the relativistic formulation de Broglie had developed."[71] However, although Einstein did not apply the details of de Broglie's theory to his work on the quantum gas, he did endorse them in some measure. In the paper in question, he reviewed de Broglie's fundamental (relativistic) arguments in favor of phase waves, as well as the result that the group velocity is equal to the velocity of the particle.[72]

In his subsequent work on the wave theory, Schrödinger also was impressed by de Broglie's result $U = v$ for wave groups (although it was not among the features of de Broglie's theory that were originally to attract his attention).[73] Schrödinger's first attempt to apply and extend de Broglie's ideas led to failure. He had tried in late 1925 to con-

struct for the atomic electron a phase wave sufficiently refracted as to make its rays trace out an elliptical Bohr–Sommerfeld orbit, but found that this led to insurmountable difficulties.[74] He then turned to the problem of the quantum gas, and showed how to apply the wave theory to calculate the frequencies of the characteristic oscillations of the gas as a whole. It is noteworthy here that although Schrödinger considered each molecule with velocity v as a " 'signal,' one might say the 'wave crest,' of a wave system whose frequency lies in the neighborhood of $\nu = mc^2(1-\beta^2)^{-1/2}/h...$ (where v plays the role of signal velocity...),"[75] the characteristic frequencies of the gas are calculated as if each molecular energy level corresponds to a single phase wave. Again, it is the phase wave, and not the group or "wave crest," that is employed in the computations.

As is well known, the result $U = v$ was to re-emerge in Schrödinger's 1926 papers on the new wave mechanics. An analysis of Schrödinger's interpretation of this result shows up (not surprisingly) some interesting differences between his and de Broglie's use of this equation. However, to show this is beyond the scope of this paper, and the subject will be treated elsewhere.[76]

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APPENDIX

"*A quantum theory of uniform rectilinear motion.* Arthur H. Compton, University of Chicago.—For uniform rectilinear motion, the quantum postulate $\int pdq = nh$ states that the momentum of a system is $p = nh/q_1$, where $q_1 = \int dq$ is the displacement required to bring the system back to its initial condition. The fact that a thing in uniform rectilinear motion repeats its initial condition at regular space intervals makes it in the general sense a train of waves, for which $\lambda = q_1$. Using Bohr's correspondence principle, each value of n is identified with the order of a harmonic component of the wave. For the nth harmonic, $\lambda_n = q_1/n$. Thus in general, for sine wave, $p = h/\lambda$. But the momentum of a wave-train of energy E and velocity v is $p = E/v$. Thus $E = hv/\lambda = h\nu$. The application of these equations to electromagnetic radiation is confirmed by the change of wave-length of X-rays when scattered and by the photo-electric effect. Thus also on the quantum theory radiation consists of trains of waves. Considering a moving diffraction grating of grating space D as a train of waves, its momentum is similarly nD/h, which is the basic hypotheses [sic] of Duane's quantum theory of diffraction."[20]

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[1]E. MacKinnon, Am. J. Phys. **44**, 1047 (1976).

[2]J. M. Espinosa, Am. J. Phys. **50**, 357 (1982).

[3]For historical details concerning the evolution of de Broglie's ideas, see R. de A. Martins and H. R. Brown (unpublished).

[4]L. de Broglie, Ann. Phys. **3**, 22 (1925). All page references to de Broglie's thesis will refer to this edition.

[5]Reference 4, p. 34.

[6]De Broglie's use of this relation to ascribe a proper frequency to the electron is not altogether new, since a similar use had already been proposed by Stark in 1907; but Stark's work had long been forgotten, and de Broglie's interpretation of $E = h\nu$ may have looked very odd to his contemporaries. For historical data, see Martins and Brown, Ref. 3.

[7]This is an aspect misunderstood by MacKinnon (Ref. 1), who states: "In the center-of-mass system there is no distinction to be drawn between particle velocity and phase velocity." This is wrong: in the center-of-mass system the phase velocity is infinite, and the particle velocity is null. Espinosa (Ref. 2) has correctly understood this point.

[8]L. de Broglie, C. R. Acad. Sci. **180**, 498 (1925).

[9]L. de Broglie, C. R. Acad. Sci. **177**, 507 (1923); **177**, 548 (1923); **177**, 811 (1923).

[10]G. Gamow, C. R. Acad. Sci. **183**, 875 (1926).

[11]For a discussion of the compatibility of wave groups with de Broglie's theory, see Sec. IV of this article.

[12]The behavior of wave packets has received much attention in the pages of this Journal: H. M. Bradford, Am. J. Phys. **44**, 1058 (1976); J. Snygg, *ibid.* **48**, 964 (1980); J. R. Klein, *ibid* **48**, 1035 (1980); J. E. Farina, *ibid.* **45**, 1200 (1977); H. C. Woodsum and K. R. Brownstein, *ibid.* **45**, 667 (1977); H. W. Lee, *ibid.* **50**, 438 (1982).

[13]There are several ways of representing this relation; we use here a simple form where only wave properties appear.

[14]We find this statement in H. A. Medicus, Phys. Today **27**, 38 (1974); E. MacKinnon, Am. J. Phys. **44**, 1047 (1976); E. MacKinnon, *ibid.* **45**, 872 (1977); J. M. Espinosa, *ibid.* **50**, 357 (1982). An author states that the relation $\lambda = h/p$ does not occur in de Broglie's thesis, although it is directly implied by the content of the thesis: P. Derek, *ibid.* **48**, 283 (1980). He has been criticized by Haslett, who remarks again that only the nonrelativistic approximation appears once in de Broglie's thesis: J. Haslett, *ibid.* **49**, 192 (1981).

[15]Reference 4, p. 22.

[16]Reference 4, p. 54.

[17]Reference 4, p. 57.

[18]Reference 4, p. 60.

[19]Reference 4, p. 61.

[20]A. H. Compton, Phys. Rev. **23**, 118 (1924).

[21]For a description of the historical context of Compton's paper, see Martins and Brown, Ref. 3.

[22]L. Brillouin, J. Phys. **6**, 337 (1925); Physica **5**, 396 (1925); Ann. Ec. Norm. Sup. **37**, 357 (1920).

[23]L. de Broglie, C. R. Acad. Sci. **177**, 507 (1923).

[24]L. de Broglie, C. R. Acad. Sci. **177**, 548 (1923). This abrupt change in de Broglie's attitude towards the physical status of the phase waves may have come about in the following way. It seems that de Broglie's recognition of the possibility of a wave-group model for the quantum also occurred between the writing of the first and second 1923 papers on phase waves. De Broglie may have recalled at this point that there were already *bona fide* examples in physics of wave groups with subluminal group velocity, whose constituent harmonic waves have velocity greater than c. Thus he may have concluded that his superluminal phase waves were not necessarily as fictitious as he had originally supposed.

[25]Reference 4, pp. 36 and 37. See also MacKinnon, Ref. 1, p. 1050, for a detailed examination of this analogy.

[26]Reference 4, pp. 38 and 39. A translation of de Broglie's introduction of wave groups in his thesis is presented in an appendix by Espinosa, Ref. 2; and the whole first chapter of de Broglie's thesis, where this part is included, has been translated by Haslett [Am. J. Phys. **40**, 1315 (1972)].

[27]Reference 4, pp. 39 and 40.

[28]See Schlegel's criticism of MacKinnon, together with a reply: R. Schlegel, Am. J. Phys. **45**, 871 (1977); E. MacKinnon, *ibid.* **45**, 872 (1977).

[29]See, e.g., A. S. Davydov, *Quantum Mechanics* (Addison-Wesley, Reading, MA, 1968), p. 8; R. M. Eisberg, *Fundamentals of Modern Physics* (Wiley, New York, 1964), pp. 141–146.

[30]See, e.g., Espinosa, Ref. 2, p. 360.

[31]See J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1967), p. 215. A possible reason as to why the existence of ΔU is so often overlooked is that in the usual derivation of U, the Taylor expansion of $\omega(k)$ around k_0,

$$\omega(k) = \omega(k_0) + (k-k_0)(d\omega/dk)_0 + (k-k_0)^2/2(d^2\omega/dk^2)_0 + \cdots,$$

is used, wherein the third term in the expansion is neglected by hypothesis. This does not mean, however, that $d^2\omega/dk^2$, and hence ΔU, vanish.

(As it happens, for de Broglie waves the cubic and further terms in the expansion do vanish in the classical limit.) We remark that one recent treatment that does explicitly calculate Δv and $d^2\omega/dk^2$ is that of E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1970), p. 25.

[32]See also Espinosa, Ref. 2, Eq. (36), p. 361.

[33]See, for example, Eisberg, Ref. 29 and Espinosa, Ref. 2, Eq. (37), p. 361.

[34]An exception is Espinosa, Ref. 2, p. 361. MacKinnon also acknowledges that the value of v is not fixed.

[35]If $c^2/V(k_0)$ is equated with v, how are we to interpret $c^2/V(k')$, where $k'(k' \neq k_0)$ is another value of k in the band Δk? It is interesting that one occasionally finds the result $U = p_0/m$, where $p_0 = \hbar k_0$. (See, for example, Davydov, Ref. 29.)

[36]Reference 1, p. 1047.

[37]Reference 4, pp. 38 and 39.

[38]Reference 1, p. 1051.

[39]Reference 2, p. 361.

[40]Reference 1, p. 1051. We may remark that MacKinnon's use of the term "defines" here is questionable. The dispersion relation is, to repeat, an outcome of de Broglie's basic postulates, and its expression in terms of the refractive index adds nothing new. An interesting analysis of some aspects of this dispersion relation is found in D. Paul, Am. J. Phys. **48**, 283 (1980).

[41]Reference 2, p. 361.

[42]Interpretation B should not be conflated, however, with the modern view. In quantum mechanics, the quantum indeterminacy in the momentum p of the system is determined by its (pure) state, whereas de Broglie makes no connection between the size of $\delta\beta$ and the dynamical state of the electron. In fact, this is not required in the derivation of $U = v$, and this is all that mattered to de Broglie (see Sec. IX for further comments in this respect).

[43]Reference 1, Secs. VI and VII.

[44]One might still want to defend the above reconstruction by saying that the phase waves are "virtual" (although their virtual existence has real, physical consequences). This kind of reasoning is found in the famous paper on virtual radiation: N. Bohr, H. Kramers, and J. Slater, Z. Phys. **24**, 69 (1924); Philos. Mag. **47**, 785 (1924). But this is merely playing with words, as Schrödinger (who was otherwise sympathetic to the paper of Bohr *et al.*) was effectively to complain (letter from Schrödinger to Bohr, 24 May 1924).

[45]The above described interpretation is similar to de Broglie's description of the electron in papers published after his thesis: L. de Broglie, J. Phys. **7**, 321 (1926); **8**, 225 (1927). The mathematical description used in these papers is compatible with de Broglie's picture of the extended quantum in his thesis. But de Broglie's presentation in the articles where he introduces the group velocity is easier to interpret if we assume an indeterministic approach; the interpretation presented here would require us to say that de Broglie made a mistake when he said that the velocity of the particle is not completely determined.

[46]A. van der Merwe, Found. Phys. **12**, 959 (1982).

[47]Reference 1, p. 1049. MacKinnon incorrectly writes [following his Eq. (12)] the de Broglie resonance condition as $\lambda = 2\pi na$, instead of $\lambda = 2\pi a/n$.

[48]Reference 4, pp. 62–65.

[49]In relation to modern discussions of de Broglie's resonance condition, two points deserve mention. The first is that in several such discussions, the resonance condition is said to give rise to a standing wave for the orbit in question. See, for example, Eisberg, Ref. 29, pp. 151 and 152; Merzbacher, Ref. 31, p. 5; and A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1970), Vol. I, p. 55. However, a standing wave has fixed nodes, and this is clearly not the case for the de Broglie traveling phase wave which obeys the resonance condition. (Moreover, a standing wave would require the superposition of two phase waves traveling in opposite directions, which is at odds with de Broglie's fundamental picture of the orbital electron.) The second point is that not all authors justify the resonance condition on the basis of the same reasoning. Frequently, the condition is related to the permanence or stability of the wave (i.e., the avoidance of destructive interference in the course of time). However, it has also been argued, incorrectly, that the single valuedness of the wave function depends on the resonance condition. See A. Sokolov, Y. Loskutov, and I. Ternov, *Quantum Mechanics* (Holt, Rinehart and Winston, New York, 1966), p. 42. These authors forget that the de Broglie waves obey the superposition principle.

[50]McKinnon, Ref. 1, p. 1053.

[51]L. de Broglie, Ref. 23. In this connection, see also de Broglie's footnote 1, p. 548, in his second 1923 paper (Ref. 24).

[52]For an alternative derivation of this result within de Broglie's theory, which is also immune to MacKinnon's criticism, see Espinosa, Ref. 2, p. 360.

[53]L. de Broglie, C. R. Acad. Sci. **179**, 676 (1924).

[54]MacKinnon, Ref. 1, p. 1047.

[55]C. Davisson and C. H. Kunsman, Science **64**, 522 (1921); Phys. Rev. **22**, 242 (1923).

[56]W. Elsasser, Naturwissenschaften **13**, 711 (1925).

[57]C. Davisson and L. H. Germer, Nature **119**, 558 (1927); Phys. Rev. **30**, 705 (1927). See also C. Davisson, J. Opt. Soc. Am. **18**, 193 (1929).

[58]A. Bethe, Naturwissenschaften **15**, 786 (1927).

[59]See G. P. Thomson, Am. J. Phys. **29**, 821 (1961).

[60]G. P. Thomson and A. Reid, Nature **119**, 890 (1927).

[61]G. P. Thomson, Proc. R. Soc. London Ser. A **117**, 600 (1927); **119**, 651 (1928).

[62]A. Reid, Proc. R. Soc. London Ser. A **119**, 663 (1928); R. Ironside, *ibid.* **119**, 668 (1928).

[63]G. P. Thomson, Proc. R. Soc. London Ser. A **117**, 600 (1927).

[64]There is a very good discussion of the early experimental tests in Jean J. Trillat, *Les Preuves Expérimentales de la Mécanique Ondulatoire* (Hermann, Paris, 1934).

[65]S. Kikuchi, Jpn. J. Phys. **5**, 83 (1928); Phys. Z. **31**, 777 (1930).

[66]E. Rupp, Ann. Phys. **9**, 458; **10**, 927 (1931); M. Kosman and A. Alichanian, Naturwissenschaften **21**, 250 (1933); Phys. Z. Sowjetunion **4**, 551 (1933).

[67]J. V. Hughes, Philos. Mag. **19**, 129 (1935).

[68]See V. Raman and P. Forman, Hist. Stud. Phys. Sci. **1**, 291 (1969) and especially L. Wessels, Stud. Hist. Philos. Sci. **10**, 311 (1977).

[69]See, for example, the comments of Rabi and Wigner in the Symposium volume: *Some Strangeness in the Proportion*..., edited by H. Woolf (Addison-Wesley, Reading, MA, 1980), pp. 471 and 472.

[70]A. Einstein, Preuss. Akad. Wiss. Berlin Ber., 3 (1925).

[71]MacKinnon, Ref. 1, p. 1053.

[72]Einstein, Ref. 70. See also Raman and Forman, Ref. 68, p. 311, footnote 60.

[73]See Wessels, Ref. 68, p. 320.

[74]Wessels, Ref. 68, p. 322.

[75]E. Schrödinger, Phys. Z. **27**, 95 (1926), as translated in Wessels, Ref. 68, pp. 324 and 325. In this otherwise excellent study of the development of Schrödinger's wave mechanics, Wessels states (p. 327) that whereas for de Broglie, "... particle and wave were separate entities... each affecting the behavior of the other... [for] Schrödinger, on the other hand, a 'particle' was only a special part of a wave (or group of waves)... What significance the equality [$U = v$] did have in de Broglie's dualistic wave-particle model is not clear. Schrödinger saw in it the basis of a pure wave theory of matter." We have seen that de Broglie did not consistently hold to a dualistic model until after the thesis. Moreover, it is one of the aims of our paper to further clarify the meaning of the result $U = v$ in de Broglie's early work.

[76]See Martins and Brown, Ref. 3.