The Origin of Dimensional Analysis

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ABSTRACT: *A paper by Macagno* (**3**) *in this Journal is discussed. The origin of the concept of physical dimensions is traced back to ideas previously used in analytic geometry. Descartes' use of the word "dimension" in the study of physical magnitudes is shown to have properties completely different from Fourier's dimensions, being therefore unimportant to the evolution of dimensional analysis. It is also shown that the principle of homogeneity was used in the derivation of physical equations sixty years prior to the publication of Fourier's work, and that the latter was aware of this paper, that may be considered the earliest publication on dimensional analysis.*

I. Introduction

The complete history of dimensional analysis has not yet been written. Several authors have at one time or another complained of this vacancy (**1**, **2**), but to this author's knowledge only one paper has been issued hitherto on this subject: Macagno's review published in this Journal (**3**). In his paper, Macagno begins by describing some general geometrical ideas about dimensions. He then refers to the rise of the conception of derived magnitudes produced by multiplication or division of other magnitudes, and arrives at Fourier's concept of physical dimensions. He then jumps to Rayleigh's and others' contributions to dimensional analysis, after 1877.

There appear to be some inaccuracies and gaps in (**3**) which we will discuss in this paper. First, there is a wide lacuna between the description of the geometrical concept of dimension and that of physical dimensions established by Fourier. Next, the introduction of Descartes as a forerunner of Fourier is not correct. And third: according to Macagno's account, dimensional analysis proper began more than fifty years after Fourier's work. Actually dimensional analysis was created *before* Fourier's concept of physical dimensions. These three corrections to Macagno's review will be briefly discussed.

II. Geometrical Dimensions

The ancient, elementary ideas about mathematical dimensions are well known. Although the basic concept was controversial, there was no doubt that lines, surfaces and solids had respectively one, two, and three dimensions (**4–6**).

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In the early development of geometry it was noticed that areas were proportional to the product of two lines, and volumes to the product of three lengths, or one length vs an area. From this geometrical property, an analogy was drawn to abstract numbers, and any number that could be expressed as a product of two (or three) natural numbers was called a rectangular (or solid) number (**5, 7**). The names "square number" and "cubic number" are vestiges of this analogy, and show that exponents were always associated with a geometrical interpretation.

In geometry, every product of two (or three) lengths was interpreted as an area (or volume). And thus, in the beginnings of analytic geometry, every term of the equations where appeared the product of two (or three) lengths was considered a bidimensional (or tridimensional) entity (**8**). From this geometric analogy, even the exponents of algebraic equations received the name of *dimensions* (**9, 10**). The association between this name and algebraic exponents was conspicuous in the work of Descartes (**11**) and that of most geometers until the 19th century. At the end of that century, this use disappeared, and in modern treatises the word "dimension" is not used anymore as synonymous of "exponent", "power", and "degree".

The mathematical concept of geometrical dimensions was related to the principle of homogeneity. According to old rules (**5**), only magnitudes of the same kind (homogeneous quantities) could be added or equalized, and only these had a (numeric) ratio. According to this principle, an area could not be added to a length or volume, and therefore any geometric equation should consist of terms representing the same kind of magnitude. If in a geometrical equation only abstract numbers and lengths appeared, then there would be an easy way of verifying whether it obeyed or not the principle of homogeneity: the algebraic sum of the exponents of the lengths in each term of the equation should be a constant number. This number was called the degree or dimension of the equation.

Sometimes it is asserted that Descartes has dropped the requirement of homogeneity of geometrical equations (**3**, **12**, **13**), and that this was a major advance in analytic geometry. Actually, Descartes explicitly stated the principle of homogeneity (**11**), and this rule was still described and used by most French textbooks in the 19th century (**14–16**).

By this series of conceptual transformations, the general principle of homogeneity of magnitudes was transformed into a rule concerning the exponents (dimensions) of some signs that entered into the equations. This was a major step preceding the creation of the concept of physical dimensions. It is also important to remark that the geometrical concept of dimension was linked to geometrical properties of similar figures: in two similar geometrical sets, if the linear dimensions were in a ratio m, then the areas would be in a ratio m^2 and the volumes in a ratio m^3 (that is, the ratio at a power corresponding to the dimensionality of surfaces and volumes). Such properties opened the possibility of creating rationalized systems of geometrical units. In antiquity, the units of length, area, and volume, were not interrelated as modern ones, because they have arisen from practical necessities (**17**). The development of pure geometry led to the creation of units such as those used in the metrical system (**18**). In the rationalized systems, the unit of area (volume) is that of a square (cube) of unit side. Accordingly, the unit of area (volume) varies as the square (cube) of the unit of length. Thus, the relations between geometrical units became the expression of the relations between the dimensions of the respective magnitudes.

III. Fourier's Physical Dimensions

The properties of geometric dimensions and the analytic use of the word "dimension" in the same sense as "exponent" were certainly known to Fourier: he uses this meaning of the word at various points in his writings (**19**). It was from this mathematical concept that Fourier drew his analogous concept of physical dimensions. Several evidences described below support this conclusion. In the *Analytic Theory of Heat,* Fourier uses the expressions "dimension" and "exponent of dimension" interchangeably (**20**), and it would be very difficult to understand this use without referring to the previous analytic use. Also, Fourier is specially concerned with the homogeneity of physical equations, and he reduces homogeneity to a property of sets of exponents associated to each kind of magnitude. Besides, his first use of this concept (**21**) was limited to the consideration of length dimensions in physical magnitudes, that is, it was a simple extension of the analytic concept to the geometric aspects of physical quantities. An explicit presentation of the main properties of Fourier's concept will show its close resemblance to the previous geometrical idea:

JBJF.1. The units of some (derived) magnitudes depend (on a unique way) on the units of other (basic) magnitudes.

JBJF.2. If the value of a physical quantity is multiplied by m^D when a unit becomes m times smaller, then D is the exponent of dimension (or dimension) of this magnitude relative to that unit.

JBJF.3. The physical equations express relations between physical quantities, and are therefore independent of the chosen units.

JBJF.4. If we replace the physical quantities of a physical equation by their values in any system of units, the equation is satisfied.

JBJF.5. In order that this property may hold, the equation must become homogeneous relative to m when the symbol of each physical quantity is replaced by m^D.

As has been shown above, all these properties were already known and used within analytic geometry before Fourier's work. Fourier's main contribution was the use of these ideas in physical contexts, and the consideration of other basic units (such as time) besides the unit of length. Fourier's extension was so natural, that some later authors did not realize its difference from purely geometrical ideas (**22, 23**).

IV. Descartes on Physical Dimensions

According to Macagno (**3**), Descartes "already used a language that conveys modern concepts" when he refers to the dimensions of physical magnitudes. It is probably true that Descartes was the first to apply the concept of dimension to physical magnitudes, but his concept was neither equivalent nor similar to Fourier's, as will be shown below. It was just a secondary development, a colateral derivation in the mainstream of dimensional concepts, and it had no influence on later developments.

Macagno's source for Descartes ideas is Dugas' *History of Mechanics* (**24**). There, Dugas describes a letter from Descartes to Mersenne, dated 12 September 1638 (**25**). Besides this source, a detailed account of Descartes' concept of physical dimensions may be found in his *Rules for the Direction of Mind* (**26**). Both of these writings have been published posthumously. This author has not been able to find any reference to this concept in Descartes' works printed during his lifetime.

According to Descartes, there are virtually infinite dimensions in each subject. A dimension of a subject is any mode relative to which this subject may be considered to be divided into equal parts. So, when a body is lifted by a force to a certain height, this process may be thought as divisible in at least two ways: we could divide the body into several parts, and lift each of these parts, by smaller forces, to the same height; or we could divide the height into several equal parts and lift the body successively to each of these points. If only these aspects are considered, this phenomenon may be considered to have two dimensions. According to one of Descartes' rules, only one or two dimensions should be considered simultaneously, even when others exist in the studied subject.

Descartes is not concerned with homogeneity of physical equations, neither discusses the distinctions and relations between basic and derived units. His concept of dimension is very similar to our modern concept of magnitude. For Descartes, speed and weight *are* dimensions, and cannot properly be said to *have* dimensions. The properties of Descartes' concept are manifest in the way he uses it, and from them we may infer that Descartes' "dimensions" have no relation to Fourier's and modern dimensional concepts.

V. The Origin of Dimensional Analysis

Let us adopt Macagno's distinction between dimensional analysis and other related subjects, such as the discussion of systems of units and the study of similarity. We may say that dimensional analysis is the method of derivation of relations between magnitudes with the use of the principle of homogeneity. In this sense, as Macagno correctly remarks, Fourier did not use dimensional analysis. It may therefore seem strange that dimensional analysis was born *before* Fourier's elucidation of the concept of physical dimensions.

The first application of dimensional analysis seems to be that due to François Daviet de Foncenex (born 1734, deceased 1799). It was published in 1761, that is, 61 years before the publication of Fourier's *Analytic Theory of Heat*. Foncenex's paper (**27**) was a study of the basic laws of mechanics. He intended to provide an *a priori* demonstration of these laws. While discussing the law of composition of forces, he introduces a dimensional argument. He considers a particular case where two equal forces F are applied to the same body, with an angle A between their directions. The value of the resultant R can only depend on the magnitudes F and A. Therefore, $R = f(F, A)$. As angles have null dimension, and R and F have the same dimensions, this relation takes necessarily the form $R = F \cdot f(A)$. This means that the resultant is proportional to the component forces, when their angle is constant. In this memoir, Foncenex also remarks that this same method could be used to derive other mechanical and geometrical propositions.

This dimensional argument is reproduced in Poisson's *Treatise on Mechanics* (**28**), without reference to its source. Poisson discusses the general meaning of the principle of homogeneity in physics, but states wrong dimensional constraints, and makes no use of the principle except in the demonstration of the law of composition of forces.

The principle of homogeneity was later used by Legendre (**29**) in the derivation of some basic geometric formulae, and in the justification of Euclid's fifth postulate. In his book, Legendre refers to Foncenex's earlier use of this method.

Was Fourier aware of these works? Probably yes. It is certain that he knew the above mentioned memoir by Foncenex: he cites it in his first published work (**19**). There appears to be no direct evidence that he also knew of Legendre's work, but this seems plausible, because Legendre's attempt at a demonstration of the postulate of parallels was widely discussed among mathematicians, at that time. Let us also note that, while introducing his concept of physical dimensions, Fourier writes: "We have introduced this consideration in the theory of heat in order to render our definitions more stable, and to be used to verify the calculations; it comes from the primordial notions about the quantities; it is for this reason that, in Geometry and Mechanics, it is equivalent to the fundamental lemmas which the Greeks have left to us without demonstration" (**20**). It is difficult to explain this sentence if we suppose that Fourier did not know the previous use of the concept of dimension in Geometry and Mechanics. But the sentence becomes clear if we assume that he knew the works of Foncenex and Legendre. These authors showed that the principle of dimensional homogeneity allowed the demonstration of some basic laws that were not demonstrated by the Greeks: the law of composition of forces, and the fifth postulate of Euclid's geometry.

VI. Concluding Remarks

In this discussion of Macagno's work (**3**), we have tried to illustrate some of the dangers of the influence of implicit assumptions in historical research. In looking for antecedents of Fourier's concept, Macagno (**3**) drew attention to earlier uses of the *word* dimension (such as Descartes'), without analysing the *properties* of the concept in order to look for the previous uses of these properties that comprise the essence of the concept.

For this reason he did not study the previous relations between the geometrical concept of dimension and geometrical units, and the relation between exponents and homogeneity.

Besides, Macagno implicitly assumed that dimensional analysis could not have arisen before the establishment of the concept of physical dimensions, and therefore only looked for instances of dimensional analysis subsequent to Fourier's work. But it is a common phenomenon in the history of science that an obscure idea is used before being explicitly defined. As has been shown, this was the case in dimensional analysis.

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