Length paradox in relativity

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The apparent self-contradictions of special relativity in thought experiments, where a body is intended to traverse a slit, do not have so simple a solution when that object is three-dimensional as when it is an idealized one- or bi-dimensional body. We use a special case of this new kind of situation in order to exemplify the general method whereby any such paradox may be analyzed and shown to be no real contradiction.

I. INTRODUCTION

A well-known paradox of special relativity[1] arises when one tries to find out whether a rod of rest length l_0 will traverse a slit of equal rest length. In the simplest cases their relative motion is uniform and they keep parallel to each other.[2] The naive analysis shows an apparent contradiction: from the rod's point of view, the slit will undergo a Lorentz contraction and, becoming smaller than the rod, will not allow it to pass; from the point of view of the slit, the rod will become smaller by Lorentz contraction, and will easily pass through the slit.

If the motion is uniform and the rod is to have any chance of passing, its velocity should be inclined relative to the slit, as shown in Fig. 1(a). In such cases, as was shown by Marx,[2] relativistic analysis should take into account a relative rotation of the body, since Lorentz effect only applies to lengths parallel to the velocity direction, as shown in Fig. 1(b). Introducing this feature, in those cases studied, it is seen that there is no real contradiction: when the analysis from one reference frame shows that the rod will not pass, the same result will follow from the analysis from other reference frames.

The paradoxes studied by Marx could not be cleared up if this rotation effect were not used. A different analysis will be necessary whenever a rotation is not relevant—when, for instance, the moving body is a sphere, instead of a rod. One such problem will be stated (Sec. II) and solved (Sec. III); next a general discussion (Sec. IV) about any kind of length paradox will be developed, and we shall show that they can always be solved within relativistic kinematics.

II. THE SPHERE AND THE RING

This new form of the length paradox was suggested by one of our students, Alfonso de Orte. A ring R and a sphere S, as shown in Fig. 2(a) of nearly equal rest radius are observed to move relative to a reference frame with perpendicular uniform velocities, in such a way that the ring moves along its axis direction; their geometrical centers have such equations of motion that they are expected to meet at a certain point O, in a future time. Will the sphere pass through the ring? It seems so, since the sphere will be contracted in the direction of its motion, and so it will not touch

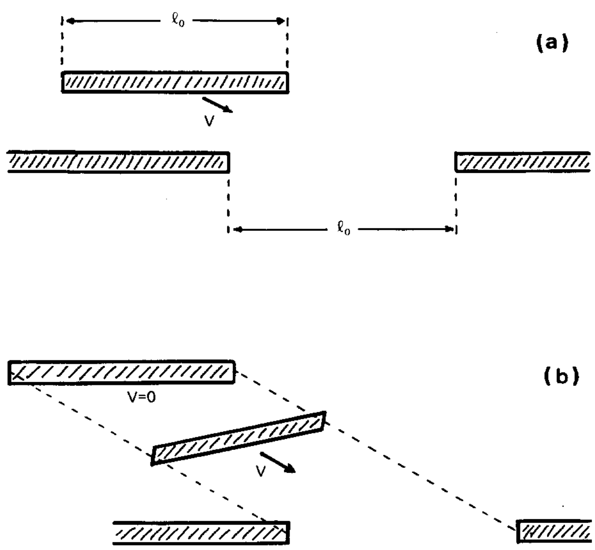

Fig. 1. (a) Length paradox: a rod moves toward a hole; both have rest length l_0. If the rod contracts, it will pass through the hole; if the hole contracts, the rod will not pass. (b) Full length of the rod does not undergo Lorentz contraction: only its projection parallel to its velocity will be contracted. This appears as a rotation of the rod, and a smaller contraction, as viewed from the hole. The rod may pass without collision if its rest length is smaller than the hole's and if widths may be neglected.

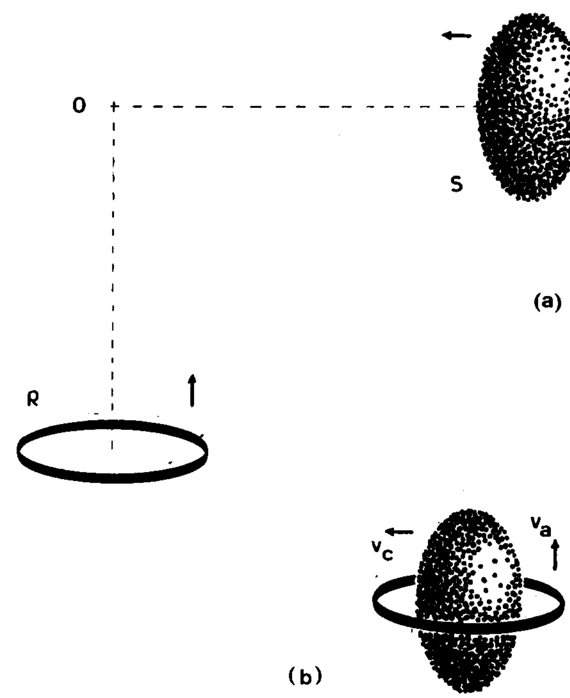

Fig. 2. Sphere (S) and ring (R) in perpendicular motions. (a) Sphere is contracted, and seems to fit inside ring (b); so it could pass through the ring.

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the ring, as shown in Fig. 2(b); but a different conclusion is reached if the problem is examined from other reference frames.

As studied from a reference frame at rest relative to the ring, the sphere moves toward it, as shown in Fig. 3(a) and is easily seen not to come from its axis direction. Now, the ring is at rest, and its diameter is $2R_0$; the sphere will undergo Lorentz contraction in the direction of its movement, but its transversal cross section will not be altered; geometrically it may be shown that it cannot pass through the ring, whatever be the value of the contraction.

From the rest frame of the sphere, the ring approaches it obliquely, and will become elliptic, with one of its dimensions shorter than $2R_0$, as shown in Fig. 3(b); the sphere can neither pass through the ring nor even stay inside it.

Two of these three analyses seem to show that there will be a collision; the conclusion of the first, on the contrary, is that collision will not happen. If quantitative analysis shows that all of them are correct, this will mean that relativity is self-contradictory, and this is not desirable; so, there must be something wrong somewhere. Our common sense tells us that the first analysis should be the wrong one and we shall try to prove it.

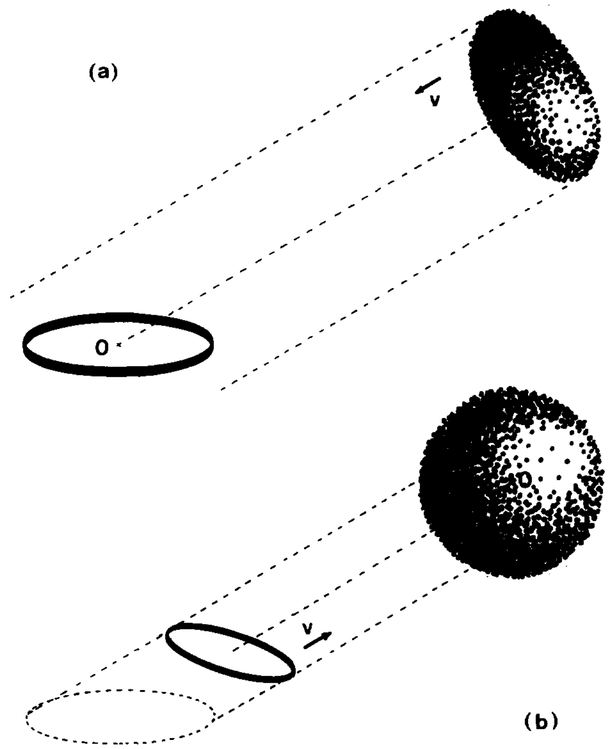

Fig. 3. (a) Sphere as seen from ring cannot pass, because it contracts only in the direction of its motion. (b) Ring as seen from sphere will not allow it to pass, because the ring transforms into an ellipse with an axis smaller than sphere's diameter.

III. COLLISION CONDITIONS

For mathematical simplicity, the problem will be reduced to a plane: we shall disregard any dimensions perpendicular to the plane of the ring's and the sphere's centers, parallel to their velocities. So the sphere will be replaced by a circle and the ring by two points. No lack of generality will follow from this simplification, as the overlooked dimensions are equal in the three frames of reference.

Let us choose a frame so that its origin lies on the point where the centers of sphere and ring will meet (Fig. 4). The x axis is parallel to the sphere's velocity, and the y axis parallel to the ring's velocity. If the time when the centers would reach O is taken to be $t = 0$, the equations of motion of the points A and B of the ring and the center C of the sphere are:

$$x_a = -R_0, \tag{1a}$$

$$y_a = v_a t, \tag{1b}$$

$$x_b = R_0, \tag{2a}$$

$$y_b = v_a t, \tag{2b}$$

$$x_c = -v_c t, \tag{3a}$$

$$y_c = 0, \tag{3b}$$

where v_a is the ring's speed, v_c the sphere's speed.

The sphere should have a radius smaller than R_0; but we shall suppose it to be exactly equal to R_0, because the difference may be as small as one wishes. If it did not contract, the equation of its circumference would be

$$(x - x_c)^2 + (y - y_c)^2 = R_0^2, \tag{4}$$

where x and y are the coordinates of any point of the circumference of the studied circle. According to relativity, all x dimensions are contracted by a factor γ given by

$$\gamma = (1 - v_c^2/c^2)^{1/2}, \tag{5}$$

where c is the speed of light in vacuum. Relative to the reference frame now used, the contracted circumference (an ellipse) will be described by relation:

$$(x - x_c)^2/\gamma^2 + (y - y_c)^2 = R_0^2. \tag{6}$$

If the sphere passes through the ring, no point of this circumference will ever meet points A and B of the ring; if there is any point of the circumference which at some time is coincident with A or B, there will be a collision of sphere and ring.

Substituting x_c and y_c from Eqs. (3a) and (3b), relation (6) becomes

$$(x + v_c t)^2/\gamma^2 + y^2 = R_0^2. \tag{7}$$

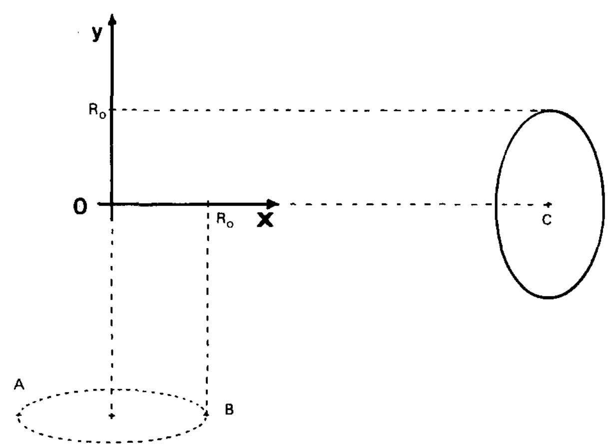

Fig. 4. A coordinate frame is chosen so that the centers of sphere and ring would meet at O; the motions of points A and B of ring and of the outer surface of sphere are studied in order to find out whether they collide or not.

If the contracted circle collides with point A, there will be a real solution to Eqs. (7), (1a), and (1b). If there is a real solution, sphere and ring do collide. Let us try it:

$$(-R_0 + v_c t)^2/\gamma^2 + (v_a t)^2 = R_0^2. \quad (8)$$

Using the value of γ given by (5), and making suitable transformations, one obtains

$$[v_c^2 + (1 - v_c^2/c^2)v_a^2]t^2 - 2R_0 v_c t + v_c^2 R_0^2/c^2 = 0. \quad (9)$$

This equation will have real solutions for t if and only if the value of Δ is non-negative, where

$$\Delta = (2R_0 v_c)^2 - 4[v_c^2 + (1 - v_c^2/c^2)v_a^2]v_c^2 R_0^2/c^2. \quad (10)$$

If sphere and ring both have speeds smaller than c (or if both are tachyons), the expression given in (10) will always be positive, as can be seen in its reduced form below:

$$\Delta = 4R_0^2 v_c^2 (1 - v_c^2/c^2)(1 - v_a^2/c^2). \quad (11)$$

In common cases (v smaller than c) there will be two solutions to Eq. (9), for t. The least solution gives the time when a point of the circle will meet point A of the ring, and so will give the time of collision between sphere and ring. It should be noted that, for any values of v_a and v_c there will be a collision, if these speeds are smaller than c. So the paradox is solved: the sphere will not pass through the ring, as analyzed in the reference frame which, qualitatively, first caused the trouble.

Perhaps there is a slip in all this: if the sphere were a little smaller than the ring, we could not write relation (6), and all that follows would be wrong. If this were the only problem, the collision would be tangential, and the solution of (9) would be unique; this would imply that Δ equals zero. This can only happen if $v_c = 0$ (the sphere at rest, with center at O) or if some of the speeds equals c. In any of these cases the sphere will pass through the ring, if it is a little smaller. But these are limiting cases which do not interest us. In any other case, a detailed analysis (omitted here) leads to the conclusion that for any values of v_a and v_c there will be a collision unless the radius difference is greater than a certain function of v_a and v_c. This means that it is not enough that the sphere be "a little bit smaller" than the ring; it should be definitely smaller in order to pass through it. In this case, analysis from the two other reference frames will show the same result.

Not everything is solved by this mathematical analysis; someone could say: "Well, you showed a fourth analysis which does give the same results as the two former ones; but you did not demonstrate that the first one was wrong, and the contradiction is not undone." This is true. We should go back to the first argument and try to find an error there.

Let us suppose that the sphere did really enter the ring, and the situation is as shown in Fig. 2(b). Can the sphere get out of the ring, or, in other words, can the ring rise so fast as not to collide with the sphere? This deserves a detailed examination. Let us first suppose that the sphere does not have a very high speed, as compared with c. Then its contracted radius

$$R = R_0(1 - v_c^2/c^2)^{1/2} \quad (12)$$

can be computed, in second-order approximation, by

$$R = R_0(1 - v_c^2/2c^2) \quad (13)$$

as can be seen expanding (12) in series. So at the instant depicted at Fig. 2(b), the distance between the ring and the sphere rim is

$$d = R_0 - R = R_0 v_c^2/2c^2 \quad (14)$$

in the second-order approximation of (13).

The sphere will travel this distance in a time

$$t = d/v_c = R_0 v_c/2c^2 \quad (15)$$

and in this time the ring will rise a distance

$$d' = v_a t = R_0 v_c v_a/2c^2. \quad (16)$$

Even if the ring rises at speed c, it can only rise to a distance $R_0 v_c/2c$; and as v_c is supposed to be small compared with c, the ring can only rise a small fraction of the radius R_0 while the sphere traverses the distance which separates them at time $t = 0$. There will be a collision. So we cannot predict qualitatively whether, starting from situation of Fig 2(b), the ring will collide with the sphere or not; the analysis presented in Sec. II was too vague to provide any definite prediction. It seems that, if the speed of the ring were very large, the sphere would not move much while the ring rises, and no collision would follow. This is a real possibility; but calculation shows that the ring must be superluminal ($v_a > c$), a condition already reached above.

It is interesting that we cannot give a clear qualitative analysis of the situation which showed that collision cannot be avoided. And in fact the sphere and ring paradox, as presented here, can only be solved in a quantitative way. We can prove this: if the contraction factor γ were not exactly the one given in (5), the values of v_a and v_c could be so chosen that a real contradiction would follow. Let us suppose that the contraction factor is given by

$$\gamma = (1 - kv_c^2/c^2)^{1/2}. \quad (17)$$

Applying this relation in (8) and going through steps analogous to (9), (10), and (11), we reach to

$$\Delta = 4R_0^2 v_c^2 (1 - kv_c^2/c^2)(1 - kv_a^2/c^2). \quad (18)$$

If k is greater than 1, Δ can have negative values for a suitable choice of v_a and v_c; and, in this case, no collision would happen between sphere and ring. The analysis from the point of view of the two other reference frames, corresponding to Fig. 3(a) and 3(b) would not be changed by this alteration of the equation of contraction; and the paradox would remain.

If the study of collision conditions depends so critically on the form of the Lorentz contraction equation, and furthermore depends on geometrical conditions, one might be tempted to find another form of the length paradox which would lead to the discovery of a real contradiction in special relativity. Now we shall prove that this will never happen.

IV. GENERAL ANALYSIS

Any modification of the length paradox must have this form:

(a) There is a relative motion between a body and a hole.

(b) The analysis relative to one reference frame S shows that the body may pass through the hole.

(c) The analysis from the point of view of another reference frame S' shows that the body cannot pass through the hole.

Someone may invent a paradox where length contraction plays an important role, and where no object is intended to pass through any hole; but the class of paradoxes which has hitherto been called "length paradox" can be described by the three above conditions. This can be called the general form of "hole paradoxes."

Any problem of this kind can be transformed to a collision problem: the two last propositions may be restated in the form:

(b′) There is no collision between the body and the contour of the hole, as analyzed from reference frame S.

(c′) There is collision between body and the contour of the hole, as analyzed from reference frame S'.

May such a paradox result from any situation studied in the context of relativity theory? Yes, if we do not develop the complete analysis, and apply just one or two theorems, instead of the whole theory; but no contradiction can occur if one applied Lorentz transformations, that is, the whole of relativistic kinematics, instead of using just the Lorentz contraction equation.

We will depart from proposition (c′), and will show that, if it is true, (b′) must be false.

If there is a collision between body and the contour of the hole, as analyzed from frame S', then, at the time of collision, there is at least one point of the body which has the same spacial coordinates as one point of the hole contour. Let us call A and B, respectively, these material points, and let the tetrad

$$e' = (x_0', y_0', z_0', t_0') \tag{19}$$

describe the event of collision of the points, relative to reference frame S'.

These space-time coordinates may be transformed to frame S, by means of Lorentz transformations, if special relativity applies to both referentials. Let the result of the transformation be

$$e = (x_0, y_0, z_0, t_0). \tag{20}$$

We want to prove that this tetrad represents a real event: the collision of the same material points A and B of body and hole contour, as viewed from frame S. But this is obvious: if the description of the motion of point A, relative to S', says that it is at point

$$P' = (x_0', y_0', z_0') \tag{21}$$

at time t_0', the description of the motion of the same point, relative to S, will associate a position

$$P = (x_0, v_0, z_0) \tag{22}$$

to this point, at time t_0, if the descriptions are to be relativistically coherent (because the same transformations which apply to event coordinates apply to point coordinates); the same may be said of point B; so, they both will be at point P at time t_0, as viewed from frame S; and there will be a collision, contradicting proposition (b′). So it is not possible that the description of two motions, relative to one frame, includes a collision and the Lorentz transform of this description to another frame does not show the same collision (although at a different time and position). So the set of propositions (a), (b′), (c′) contradicts relativistic kinematics, and such a paradox cannot happen in relativity. In fact, this does not even depend on the form of the Lorentz transformations: any one-to-one transformation of space-time coordinates will prohibit the contradiction. So there cannot be a length contradiction in general relativity, either, as it is based on coordinate transformation.[3]

Could there be a contradiction if the Lorentz contraction equation was used, instead of Lorentz transformations? Yes. This happens if the analysis omits the correct time transformations; if the fourth coordinate is not forgotten, no contradiction may arise, because the Lorentz contraction equation is derivable as a special case of Lorentz transformations, and so cannot contradict them.

Whoever does not believe this should try to find a new length or hole paradox which shows a contradiction in relativity theory. We predict that such a paradox will be solved—not because we have a blind faith in relativity, but because the analysis of its structure shows this. We do not ask everyone to stop thinking about such paradoxes. Although they will never undermine relativity, they may be didactly very instructive, and are worth studying.

V. SUMMARY

We have described a new thought experiment where a qualitative analysis using Lorentz contraction and relative rotation do not solve an apparent contradiction. The quantitative analysis proves that there is no real contradiction, but also shows that, if the contraction obeyed a slightly different law there would be a contradiction. A general analysis then proves that any theory which includes a one-to-one space-time coordinate transformation will never be subjected to failure by any form of the length (or hole) paradox; and so Lorentz transformations (not Lorentz contraction) must, in general, be used to assure that special relativity is length-paradox proof.

[1]The paradox has already been used as an April-first joke, by M. Gardner, Sci. Am. **232**, 127(1975), and has appeared in E. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, San Francisco, 1966).

[2]The original paradox was first stated by W. Rindler, Am. J. Phys. **29**, 365(1961) and involved an analysis of the acceleration of the rod, which then became curved as seen from one reference frame; uniform transverse motion was studied by R. Shaw, Am. J. Phys. **30**, 72(1962) and by E. Marx, Am. J. Phys. **35**, 1127 (1967).

[3]Einstein was well aware that the study of lengths and time intervals was liable to great difficulties, but which no contradictions would arise in a theory which was founded on coordinates and coordinate transformation; see his analysis of covariance and of reduction of physics to point events in A. Einstein, "The Foundation of the General Theory of Relativity," printed as a chapter of *The Principle of Relativity* (Dover, New York, 1923), especially Chap. 3, p. 115.